

REALIZABLE

hypothesis space \mathcal{H} to allow polynomials over both x and $\sin(x)$, and find that the data in (c) can be fitted exactly by a simple function of the form $ax + b + c \sin(x)$. This shows the importance of the choice of hypothesis space. We say that a learning problem is **realizable** if the hypothesis space contains the true function. Unfortunately, we cannot always tell whether a given learning problem is realizable, because the true function is not known.

In some cases, an analyst looking at a problem is willing to make more fine-grained distinctions about the hypothesis space, to say—even before seeing any data—not just that a hypothesis is possible or impossible, but rather how probable it is. Supervised learning can be done by choosing the hypothesis h^* that is most probable given the data:

$$h^* = \operatorname{argmax}_{h \in \mathcal{H}} P(h|data) .$$

By Bayes' rule this is equivalent to

$$h^* = \operatorname{argmax}_{h \in \mathcal{H}} P(data|h) P(h) .$$

Then we can say that the prior probability $P(h)$ is high for a degree-1 or -2 polynomial, lower for a degree-7 polynomial, and especially low for degree-7 polynomials with large, sharp spikes as in Figure 18.1(b). We allow unusual-looking functions when the data say we really need them, but we discourage them by giving them a low prior probability.



Why not let \mathcal{H} be the class of all Java programs, or Turing machines? After all, every computable function can be represented by some Turing machine, and that is the best we can do. One problem with this idea is that it does not take into account the computational complexity of learning. *There is a tradeoff between the expressiveness of a hypothesis space and the complexity of finding a good hypothesis within that space.* For example, fitting a straight line to data is an easy computation; fitting high-degree polynomials is somewhat harder; and fitting Turing machines is in general undecidable. A second reason to prefer simple hypothesis spaces is that presumably we will want to use h after we have learned it, and computing $h(x)$ when h is a linear function is guaranteed to be fast, while computing an arbitrary Turing machine program is not even guaranteed to terminate. For these reasons, most work on learning has focused on simple representations.

We will see that the expressiveness–complexity tradeoff is not as simple as it first seems: it is often the case, as we saw with first-order logic in Chapter 8, that an expressive language makes it possible for a *simple* hypothesis to fit the data, whereas restricting the expressiveness of the language means that any consistent hypothesis must be very complex. For example, the rules of chess can be written in a page or two of first-order logic, but require thousands of pages when written in propositional logic.

18.3 LEARNING DECISION TREES

Decision tree induction is one of the simplest and yet most successful forms of machine learning. We first describe the representation—the hypothesis space—and then show how to learn a good hypothesis.

18.3.1 The decision tree representation

DECISION TREE

A **decision tree** represents a function that takes as input a vector of attribute values and returns a “decision”—a single output value. The input and output values can be discrete or continuous. For now we will concentrate on problems where the inputs have discrete values and the output has exactly two possible values; this is Boolean classification, where each example input will be classified as true (a **positive** example) or false (a **negative** example).

POSITIVE

NEGATIVE

A decision tree reaches its decision by performing a sequence of tests. Each internal node in the tree corresponds to a test of the value of one of the input attributes, A_i , and the branches from the node are labeled with the possible values of the attribute, $A_i = v_{ik}$. Each leaf node in the tree specifies a value to be returned by the function. The decision tree representation is natural for humans; indeed, many “How To” manuals (e.g., for car repair) are written entirely as a single decision tree stretching over hundreds of pages.

GOAL PREDICATE

As an example, we will build a decision tree to decide whether to wait for a table at a restaurant. The aim here is to learn a definition for the **goal predicate** *WillWait*. First we list the attributes that we will consider as part of the input:

1. *Alternate*: whether there is a suitable alternative restaurant nearby.
2. *Bar*: whether the restaurant has a comfortable bar area to wait in.
3. *Fri/Sat*: true on Fridays and Saturdays.
4. *Hungry*: whether we are hungry.
5. *Patrons*: how many people are in the restaurant (values are *None*, *Some*, and *Full*).
6. *Price*: the restaurant’s price range (\$, \$\$, \$\$\$).
7. *Raining*: whether it is raining outside.
8. *Reservation*: whether we made a reservation.
9. *Type*: the kind of restaurant (French, Italian, Thai, or burger).
10. *WaitEstimate*: the wait estimated by the host (0–10 minutes, 10–30, 30–60, or >60).

Note that every variable has a small set of possible values; the value of *WaitEstimate*, for example, is not an integer, rather it is one of the four discrete values 0–10, 10–30, 30–60, or >60. The decision tree usually used by one of us (SR) for this domain is shown in Figure 18.2. Notice that the tree ignores the *Price* and *Type* attributes. Examples are processed by the tree starting at the root and following the appropriate branch until a leaf is reached. For instance, an example with *Patrons* = *Full* and *WaitEstimate* = 0–10 will be classified as positive (i.e., yes, we will wait for a table).

18.3.2 Expressiveness of decision trees

A Boolean decision tree is logically equivalent to the assertion that the goal attribute is true if and only if the input attributes satisfy one of the paths leading to a leaf with value *true*. Writing this out in propositional logic, we have

$$Goal \Leftrightarrow (Path_1 \vee Path_2 \vee \dots),$$

where each *Path* is a conjunction of attribute-value tests required to follow that path. Thus, the whole expression is equivalent to disjunctive normal form (see page 283), which means

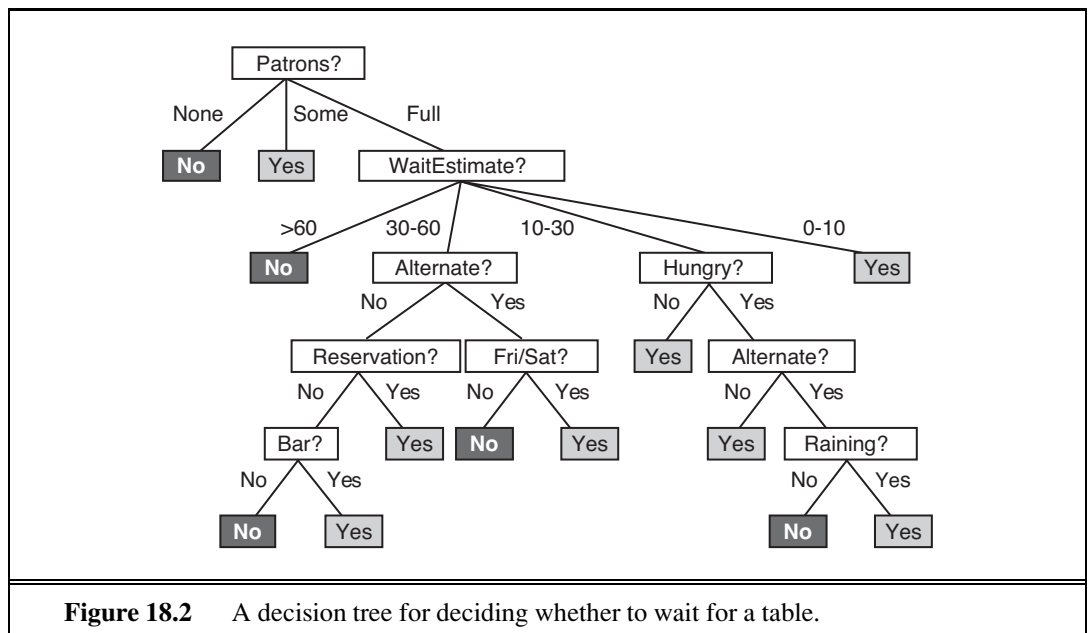
that any function in propositional logic can be expressed as a decision tree. As an example, the rightmost path in Figure 18.2 is

$$\text{Path} = (\text{Patrons} = \text{Full} \wedge \text{WaitEstimate} = 0-10) .$$

For a wide variety of problems, the decision tree format yields a nice, concise result. But some functions cannot be represented concisely. For example, the majority function, which returns true if and only if more than half of the inputs are true, requires an exponentially large decision tree. In other words, decision trees are good for some kinds of functions and bad for others. Is there *any* kind of representation that is efficient for *all* kinds of functions? Unfortunately, the answer is no. We can show this in a general way. Consider the set of all Boolean functions on n attributes. How many different functions are in this set? This is just the number of different truth tables that we can write down, because the function is defined by its truth table. A truth table over n attributes has 2^n rows, one for each combination of values of the attributes. We can consider the “answer” column of the table as a 2^n -bit number that defines the function. That means there are 2^{2^n} different functions (and there will be more than that number of trees, since more than one tree can compute the same function). This is a scary number. For example, with just the ten Boolean attributes of our restaurant problem there are 2^{1024} or about 10^{308} different functions to choose from, and for 20 attributes there are over $10^{300,000}$. We will need some ingenious algorithms to find good hypotheses in such a large space.

18.3.3 Inducing decision trees from examples

An example for a Boolean decision tree consists of an (\mathbf{x}, y) pair, where \mathbf{x} is a vector of values for the input attributes, and y is a single Boolean output value. A training set of 12 examples



Example	Input Attributes										Goal
	<i>Alt</i>	<i>Bar</i>	<i>Fri</i>	<i>Hun</i>	<i>Pat</i>	<i>Price</i>	<i>Rain</i>	<i>Res</i>	<i>Type</i>	<i>Est</i>	<i>WillWait</i>
\mathbf{x}_1	<i>Yes</i>	<i>No</i>	<i>No</i>	<i>Yes</i>	<i>Some</i>	<i>\$\$\$</i>	<i>No</i>	<i>Yes</i>	<i>French</i>	<i>0-10</i>	$y_1 = \text{Yes}$
\mathbf{x}_2	<i>Yes</i>	<i>No</i>	<i>No</i>	<i>Yes</i>	<i>Full</i>	<i>\$</i>	<i>No</i>	<i>No</i>	<i>Thai</i>	<i>30-60</i>	$y_2 = \text{No}$
\mathbf{x}_3	<i>No</i>	<i>Yes</i>	<i>No</i>	<i>No</i>	<i>Some</i>	<i>\$</i>	<i>No</i>	<i>No</i>	<i>Burger</i>	<i>0-10</i>	$y_3 = \text{Yes}$
\mathbf{x}_4	<i>Yes</i>	<i>No</i>	<i>Yes</i>	<i>Yes</i>	<i>Full</i>	<i>\$</i>	<i>Yes</i>	<i>No</i>	<i>Thai</i>	<i>10-30</i>	$y_4 = \text{Yes}$
\mathbf{x}_5	<i>Yes</i>	<i>No</i>	<i>Yes</i>	<i>No</i>	<i>Full</i>	<i>\$\$\$</i>	<i>No</i>	<i>Yes</i>	<i>French</i>	<i>>60</i>	$y_5 = \text{No}$
\mathbf{x}_6	<i>No</i>	<i>Yes</i>	<i>No</i>	<i>Yes</i>	<i>Some</i>	<i>\$\$</i>	<i>Yes</i>	<i>Yes</i>	<i>Italian</i>	<i>0-10</i>	$y_6 = \text{Yes}$
\mathbf{x}_7	<i>No</i>	<i>Yes</i>	<i>No</i>	<i>No</i>	<i>None</i>	<i>\$</i>	<i>Yes</i>	<i>No</i>	<i>Burger</i>	<i>0-10</i>	$y_7 = \text{No}$
\mathbf{x}_8	<i>No</i>	<i>No</i>	<i>No</i>	<i>Yes</i>	<i>Some</i>	<i>\$\$</i>	<i>Yes</i>	<i>Yes</i>	<i>Thai</i>	<i>0-10</i>	$y_8 = \text{Yes}$
\mathbf{x}_9	<i>No</i>	<i>Yes</i>	<i>Yes</i>	<i>No</i>	<i>Full</i>	<i>\$</i>	<i>Yes</i>	<i>No</i>	<i>Burger</i>	<i>>60</i>	$y_9 = \text{No}$
\mathbf{x}_{10}	<i>Yes</i>	<i>Yes</i>	<i>Yes</i>	<i>Yes</i>	<i>Full</i>	<i>\$\$\$</i>	<i>No</i>	<i>Yes</i>	<i>Italian</i>	<i>10-30</i>	$y_{10} = \text{No}$
\mathbf{x}_{11}	<i>No</i>	<i>No</i>	<i>No</i>	<i>No</i>	<i>None</i>	<i>\$</i>	<i>No</i>	<i>No</i>	<i>Thai</i>	<i>0-10</i>	$y_{11} = \text{No}$
\mathbf{x}_{12}	<i>Yes</i>	<i>Yes</i>	<i>Yes</i>	<i>Yes</i>	<i>Full</i>	<i>\$</i>	<i>No</i>	<i>No</i>	<i>Burger</i>	<i>30-60</i>	$y_{12} = \text{Yes}$

Figure 18.3 Examples for the restaurant domain.

is shown in Figure 18.3. The positive examples are the ones in which the goal *WillWait* is true ($\mathbf{x}_1, \mathbf{x}_3, \dots$); the negative examples are the ones in which it is false ($\mathbf{x}_2, \mathbf{x}_5, \dots$).

We want a tree that is consistent with the examples and is as small as possible. Unfortunately, no matter how we measure size, it is an intractable problem to find the smallest consistent tree; there is no way to efficiently search through the 2^{2^n} trees. With some simple heuristics, however, we can find a good approximate solution: a small (but not smallest) consistent tree. The DECISION-TREE-LEARNING algorithm adopts a greedy divide-and-conquer strategy: always test the most important attribute first. This test divides the problem up into smaller subproblems that can then be solved recursively. By “most important attribute,” we mean the one that makes the most difference to the classification of an example. That way, we hope to get to the correct classification with a small number of tests, meaning that all paths in the tree will be short and the tree as a whole will be shallow.

Figure 18.4(a) shows that *Type* is a poor attribute, because it leaves us with four possible outcomes, each of which has the same number of positive as negative examples. On the other hand, in (b) we see that *Patrons* is a fairly important attribute, because if the value is *None* or *Some*, then we are left with example sets for which we can answer definitively (*No* and *Yes*, respectively). If the value is *Full*, we are left with a mixed set of examples. In general, after the first attribute test splits up the examples, each outcome is a new decision tree learning problem in itself, with fewer examples and one less attribute. There are four cases to consider for these recursive problems:

1. If the remaining examples are all positive (or all negative), then we are done: we can answer *Yes* or *No*. Figure 18.4(b) shows examples of this happening in the *None* and *Some* branches.
2. If there are some positive and some negative examples, then choose the best attribute to split them. Figure 18.4(b) shows *Hungry* being used to split the remaining examples.
3. If there are no examples left, it means that no example has been observed for this com-

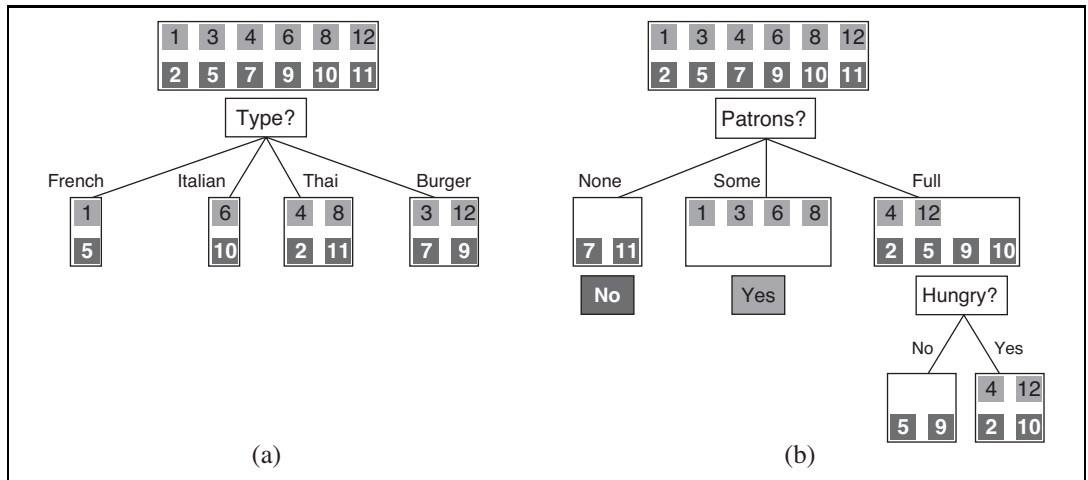


Figure 18.4 Splitting the examples by testing on attributes. At each node we show the positive (light boxes) and negative (dark boxes) examples remaining. (a) Splitting on *Type* brings us no nearer to distinguishing between positive and negative examples. (b) Splitting on *Patrons* does a good job of separating positive and negative examples. After splitting on *Patrons*, *Hungry* is a fairly good second test.

bination of attribute values, and we return a default value calculated from the plurality classification of all the examples that were used in constructing the node's parent. These are passed along in the variable *parent_examples*.

4. If there are no attributes left, but both positive and negative examples, it means that these examples have exactly the same description, but different classifications. This can happen because there is an error or **noise** in the data; because the domain is nondeterministic; or because we can't observe an attribute that would distinguish the examples. The best we can do is return the plurality classification of the remaining examples.

NOISE

The DECISION-TREE-LEARNING algorithm is shown in Figure 18.5. Note that the set of examples is crucial for *constructing* the tree, but nowhere do the examples appear in the tree itself. A tree consists of just tests on attributes in the interior nodes, values of attributes on the branches, and output values on the leaf nodes. The details of the IMPORTANCE function are given in Section 18.3.4. The output of the learning algorithm on our sample training set is shown in Figure 18.6. The tree is clearly different from the original tree shown in Figure 18.2. One might conclude that the learning algorithm is not doing a very good job of learning the correct function. This would be the wrong conclusion to draw, however. The learning algorithm looks at the *examples*, not at the correct function, and in fact, its hypothesis (see Figure 18.6) not only is consistent with all the examples, but is considerably simpler than the original tree! The learning algorithm has no reason to include tests for *Raining* and *Reservation*, because it can classify all the examples without them. It has also detected an interesting and previously unsuspected pattern: the first author will wait for Thai food on weekends. It is also bound to make some mistakes for cases where it has seen no examples. For example, it has never seen a case where the wait is 0–10 minutes but the restaurant is full.

```

function DECISION-TREE-LEARNING(examples, attributes, parent_examples) returns
a tree

if examples is empty then return PLURALITY-VALUE(parent_examples)
else if all examples have the same classification then return the classification
else if attributes is empty then return PLURALITY-VALUE(examples)
else
   $A \leftarrow \operatorname{argmax}_{a \in \text{attributes}} \text{IMPORTANCE}(a, \text{examples})$ 
  tree  $\leftarrow$  a new decision tree with root test A
  for each value  $v_k$  of A do
     $\text{exs} \leftarrow \{e : e \in \text{examples} \text{ and } e.A = v_k\}$ 
    subtree  $\leftarrow$  DECISION-TREE-LEARNING(exs, attributes - A, examples)
    add a branch to tree with label ( $A = v_k$ ) and subtree subtree
  return tree

```

Figure 18.5 The decision-tree learning algorithm. The function IMPORTANCE is described in Section 18.3.4. The function PLURALITY-VALUE selects the most common output value among a set of examples, breaking ties randomly.

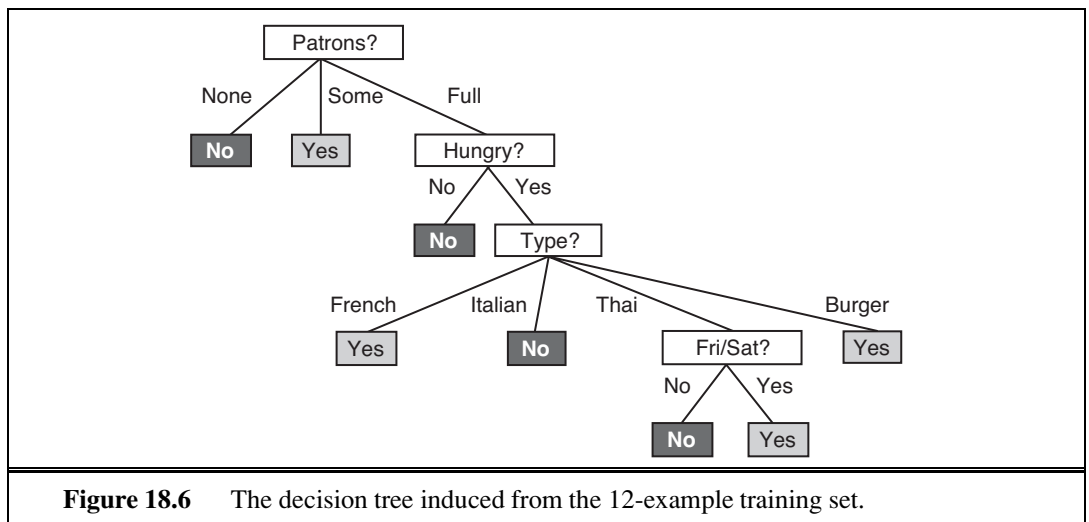
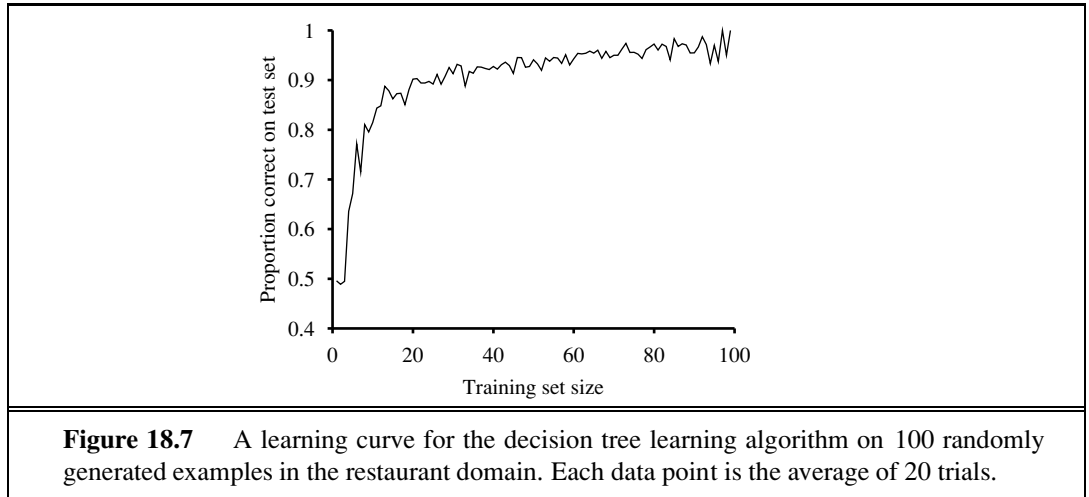


Figure 18.6 The decision tree induced from the 12-example training set.

In that case it says not to wait when *Hungry* is false, but I (SR) would certainly wait. With more training examples the learning program could correct this mistake.

We note there is a danger of over-interpreting the tree that the algorithm selects. When there are several variables of similar importance, the choice between them is somewhat arbitrary: with slightly different input examples, a different variable would be chosen to split on first, and the whole tree would look completely different. The function computed by the tree would still be similar, but the structure of the tree can vary widely.

We can evaluate the accuracy of a learning algorithm with a **learning curve**, as shown in Figure 18.7. We have 100 examples at our disposal, which we split into a training set and



a test set. We learn a hypothesis h with the training set and measure its accuracy with the test set. We do this starting with a training set of size 1 and increasing one at a time up to size 99. For each size we actually repeat the process of randomly splitting 20 times, and average the results of the 20 trials. The curve shows that as the training set size grows, the accuracy increases. (For this reason, learning curves are also called **happy graphs**.) In this graph we reach 95% accuracy, and it looks like the curve might continue to increase with more data.

18.3.4 Choosing attribute tests

The greedy search used in decision tree learning is designed to approximately minimize the depth of the final tree. The idea is to pick the attribute that goes as far as possible toward providing an exact classification of the examples. A perfect attribute divides the examples into sets, each of which are all positive or all negative and thus will be leaves of the tree. The *Patrons* attribute is not perfect, but it is fairly good. A really useless attribute, such as *Type*, leaves the example sets with roughly the same proportion of positive and negative examples as the original set.

All we need, then, is a formal measure of “fairly good” and “really useless” and we can implement the IMPORTANCE function of Figure 18.5. We will use the notion of information gain, which is defined in terms of **entropy**, the fundamental quantity in information theory (Shannon and Weaver, 1949).

ENTROPY

Entropy is a measure of the uncertainty of a random variable; acquisition of information corresponds to a reduction in entropy. A random variable with only one value—a coin that always comes up heads—has no uncertainty and thus its entropy is defined as zero; thus, we gain no information by observing its value. A flip of a fair coin is equally likely to come up heads or tails, 0 or 1, and we will soon show that this counts as “1 bit” of entropy. The roll of a fair *four*-sided die has 2 bits of entropy, because it takes two bits to describe one of four equally probable choices. Now consider an unfair coin that comes up heads 99% of the time. Intuitively, this coin has less uncertainty than the fair coin—if we guess heads we’ll be wrong only 1% of the time—so we would like it to have an entropy measure that is close to zero, but

positive. In general, the entropy of a random variable V with values v_k , each with probability $P(v_k)$, is defined as

$$\text{Entropy: } H(V) = \sum_k P(v_k) \log_2 \frac{1}{P(v_k)} = - \sum_k P(v_k) \log_2 P(v_k) .$$

We can check that the entropy of a fair coin flip is indeed 1 bit:

$$H(\text{Fair}) = -(0.5 \log_2 0.5 + 0.5 \log_2 0.5) = 1 .$$

If the coin is loaded to give 99% heads, we get

$$H(\text{Loaded}) = -(0.99 \log_2 0.99 + 0.01 \log_2 0.01) \approx 0.08 \text{ bits.}$$

It will help to define $B(q)$ as the entropy of a Boolean random variable that is true with probability q :

$$B(q) = -(q \log_2 q + (1 - q) \log_2 (1 - q)) .$$

Thus, $H(\text{Loaded}) = B(0.99) \approx 0.08$. Now let's get back to decision tree learning. If a training set contains p positive examples and n negative examples, then the entropy of the goal attribute on the whole set is

$$H(\text{Goal}) = B\left(\frac{p}{p+n}\right) .$$

The restaurant training set in Figure 18.3 has $p = n = 6$, so the corresponding entropy is $B(0.5)$ or exactly 1 bit. A test on a single attribute A might give us only part of this 1 bit. We can measure exactly how much by looking at the entropy remaining *after* the attribute test.

An attribute A with d distinct values divides the training set E into subsets E_1, \dots, E_d . Each subset E_k has p_k positive examples and n_k negative examples, so if we go along that branch, we will need an additional $B(p_k/(p_k + n_k))$ bits of information to answer the question. A randomly chosen example from the training set has the k th value for the attribute with probability $(p_k + n_k)/(p + n)$, so the expected entropy remaining after testing attribute A is

$$\text{Remainder}(A) = \sum_{k=1}^d \frac{p_k + n_k}{p + n} B\left(\frac{p_k}{p_k + n_k}\right) .$$

INFORMATION GAIN

The **information gain** from the attribute test on A is the expected reduction in entropy:

$$\text{Gain}(A) = B\left(\frac{p}{p+n}\right) - \text{Remainder}(A) .$$

In fact $\text{Gain}(A)$ is just what we need to implement the IMPORTANCE function. Returning to the attributes considered in Figure 18.4, we have

$$\text{Gain}(\text{Patrons}) = 1 - \left[\frac{2}{12} B\left(\frac{0}{2}\right) + \frac{4}{12} B\left(\frac{4}{4}\right) + \frac{6}{12} B\left(\frac{2}{6}\right) \right] \approx 0.541 \text{ bits,}$$

$$\text{Gain}(\text{Type}) = 1 - \left[\frac{2}{12} B\left(\frac{1}{2}\right) + \frac{2}{12} B\left(\frac{1}{2}\right) + \frac{4}{12} B\left(\frac{2}{4}\right) + \frac{4}{12} B\left(\frac{2}{4}\right) \right] = 0 \text{ bits,}$$

confirming our intuition that *Patrons* is a better attribute to split on. In fact, *Patrons* has the maximum gain of any of the attributes and would be chosen by the decision-tree learning algorithm as the root.

18.3.5 Generalization and overfitting

On some problems, the DECISION-TREE-LEARNING algorithm will generate a large tree when there is actually no pattern to be found. Consider the problem of trying to predict whether the roll of a die will come up as 6 or not. Suppose that experiments are carried out with various dice and that the attributes describing each training example include the color of the die, its weight, the time when the roll was done, and whether the experimenters had their fingers crossed. If the dice are fair, the right thing to learn is a tree with a single node that says “no.” But the DECISION-TREE-LEARNING algorithm will seize on any pattern it can find in the input. If it turns out that there are 2 rolls of a 7-gram blue die with fingers crossed and they both come out 6, then the algorithm may construct a path that predicts 6 in that case. This problem is called **overfitting**. A general phenomenon, overfitting occurs with all types of learners, even when the target function is not at all random. In Figure 18.1(b) and (c), we saw polynomial functions overfitting the data. Overfitting becomes more likely as the hypothesis space and the number of input attributes grows, and less likely as we increase the number of training examples.

OVERFITTING

DECISION TREE
PRUNING

For decision trees, a technique called **decision tree pruning** combats overfitting. Pruning works by eliminating nodes that are not clearly relevant. We start with a full tree, as generated by DECISION-TREE-LEARNING. We then look at a test node that has only leaf nodes as descendants. If the test appears to be irrelevant—detecting only noise in the data—then we eliminate the test, replacing it with a leaf node. We repeat this process, considering each test with only leaf descendants, until each one has either been pruned or accepted as is.

The question is, how do we detect that a node is testing an irrelevant attribute? Suppose we are at a node consisting of p positive and n negative examples. If the attribute is irrelevant, we would expect that it would split the examples into subsets that each have roughly the same proportion of positive examples as the whole set, $p/(p+n)$, and so the information gain will be close to zero.² Thus, the information gain is a good clue to irrelevance. Now the question is, how large a gain should we require in order to split on a particular attribute?

SIGNIFICANCE TEST

NULL HYPOTHESIS

We can answer this question by using a statistical **significance test**. Such a test begins by assuming that there is no underlying pattern (the so-called **null hypothesis**). Then the actual data are analyzed to calculate the extent to which they deviate from a perfect absence of pattern. If the degree of deviation is statistically unlikely (usually taken to mean a 5% probability or less), then that is considered to be good evidence for the presence of a significant pattern in the data. The probabilities are calculated from standard distributions of the amount of deviation one would expect to see in random sampling.

In this case, the null hypothesis is that the attribute is irrelevant and, hence, that the information gain for an infinitely large sample would be zero. We need to calculate the probability that, under the null hypothesis, a sample of size $v = n + p$ would exhibit the observed deviation from the expected distribution of positive and negative examples. We can measure the deviation by comparing the actual numbers of positive and negative examples in

² The gain will be strictly positive except for the unlikely case where all the proportions are *exactly* the same. (See Exercise 18.5.)

each subset, p_k and n_k , with the expected numbers, \hat{p}_k and \hat{n}_k , assuming true irrelevance:

$$\hat{p}_k = p \times \frac{p_k + n_k}{p + n} \quad \hat{n}_k = n \times \frac{p_k + n_k}{p + n} .$$

A convenient measure of the total deviation is given by

$$\Delta = \sum_{k=1}^d \frac{(p_k - \hat{p}_k)^2}{\hat{p}_k} + \frac{(n_k - \hat{n}_k)^2}{\hat{n}_k} .$$

Under the null hypothesis, the value of Δ is distributed according to the χ^2 (chi-squared) distribution with $v - 1$ degrees of freedom. We can use a χ^2 table or a standard statistical library routine to see if a particular Δ value confirms or rejects the null hypothesis. For example, consider the restaurant type attribute, with four values and thus three degrees of freedom. A value of $\Delta = 7.82$ or more would reject the null hypothesis at the 5% level (and a value of $\Delta = 11.35$ or more would reject at the 1% level). Exercise 18.8 asks you to extend the DECISION-TREE-LEARNING algorithm to implement this form of pruning, which is known as χ^2 **pruning**.

χ^2 PRUNING

With pruning, noise in the examples can be tolerated. Errors in the example's label (e.g., an example (\mathbf{x}, Yes) that should be (\mathbf{x}, No)) give a linear increase in prediction error, whereas errors in the descriptions of examples (e.g., $\text{Price} = \$$ when it was actually $\text{Price} = \$\$$) have an asymptotic effect that gets worse as the tree shrinks down to smaller sets. Pruned trees perform significantly better than unpruned trees when the data contain a large amount of noise. Also, the pruned trees are often much smaller and hence easier to understand.

One final warning: You might think that χ^2 pruning and information gain look similar, so why not combine them using an approach called **early stopping**—have the decision tree algorithm stop generating nodes when there is no good attribute to split on, rather than going to all the trouble of generating nodes and then pruning them away. The problem with early stopping is that it stops us from recognizing situations where there is no one good attribute, but there are combinations of attributes that are informative. For example, consider the XOR function of two binary attributes. If there are roughly equal number of examples for all four combinations of input values, then neither attribute will be informative, yet the correct thing to do is to split on one of the attributes (it doesn't matter which one), and then at the second level we will get splits that are informative. Early stopping would miss this, but generate-and-then-prune handles it correctly.

EARLY STOPPING

18.3.6 Broadening the applicability of decision trees

In order to extend decision tree induction to a wider variety of problems, a number of issues must be addressed. We will briefly mention several, suggesting that a full understanding is best obtained by doing the associated exercises:

- **Missing data:** In many domains, not all the attribute values will be known for every example. The values might have gone unrecorded, or they might be too expensive to obtain. This gives rise to two problems: First, given a complete decision tree, how should one classify an example that is missing one of the test attributes? Second, how

should one modify the information-gain formula when some examples have unknown values for the attribute? These questions are addressed in Exercise 18.9.

GAIN RATIO

- **Multivalued attributes:** When an attribute has many possible values, the information gain measure gives an inappropriate indication of the attribute's usefulness. In the extreme case, an attribute such as *ExactTime* has a different value for every example, which means each subset of examples is a singleton with a unique classification, and the information gain measure would have its highest value for this attribute. But choosing this split first is unlikely to yield the best tree. One solution is to use the **gain ratio** (Exercise 18.10). Another possibility is to allow a Boolean test of the form $A = v_k$, that is, picking out just one of the possible values for an attribute, leaving the remaining values to possibly be tested later in the tree.

SPLIT POINT

- **Continuous and integer-valued input attributes:** Continuous or integer-valued attributes such as *Height* and *Weight*, have an infinite set of possible values. Rather than generate infinitely many branches, decision-tree learning algorithms typically find the **split point** that gives the highest information gain. For example, at a given node in the tree, it might be the case that testing on $Weight > 160$ gives the most information. Efficient methods exist for finding good split points: start by sorting the values of the attribute, and then consider only split points that are between two examples in sorted order that have different classifications, while keeping track of the running totals of positive and negative examples on each side of the split point. Splitting is the most expensive part of real-world decision tree learning applications.

REGRESSION TREE

- **Continuous-valued output attributes:** If we are trying to predict a numerical output value, such as the price of an apartment, then we need a **regression tree** rather than a classification tree. A regression tree has at each leaf a linear function of some subset of numerical attributes, rather than a single value. For example, the branch for two-bedroom apartments might end with a linear function of square footage, number of bathrooms, and average income for the neighborhood. The learning algorithm must decide when to stop splitting and begin applying linear regression (see Section 18.6) over the attributes.

A decision-tree learning system for real-world applications must be able to handle all of these problems. Handling continuous-valued variables is especially important, because both physical and financial processes provide numerical data. Several commercial packages have been built that meet these criteria, and they have been used to develop thousands of fielded systems. In many areas of industry and commerce, decision trees are usually the first method tried when a classification method is to be extracted from a data set. One important property of decision trees is that it is possible for a human to understand the reason for the output of the learning algorithm. (Indeed, this is a *legal requirement* for financial decisions that are subject to anti-discrimination laws.) This is a property not shared by some other representations, such as neural networks.